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Remarks on Random Evolutions in Hamiltonian Representation

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Abstract

Abstract telegrapher's equations and some random walks of Poisson type are shown to fit into the framework of the Hamiltonian formalism after an appropriate time-dependent rescaling of the basic variables has been made.

§ 1. Introduction

Time evolution of random processes differs in one essential respect from evolution of conservative systems in general and Hamiltonian systems in particular. As a great number of various limit theorems attest, the final states loose all the information about the initial conditions. Thus, there is nothing to be conserved, no constants of motion can exist, and no use can be made of the powerful machine of the Hamiltonian formalism. Or so it seems. Sometimes there *are* things which do not change during time evolution, such as rates of decay and other universal exponents. This suggests that one may hope to find constants of motion and even Hamiltonian forms in at least some probabilistic systems provided one is willing to make time-dependent rescalings of the basic dynamical variables. Besides, there is something like a precendent in the history of attempts to quantize dissipative systems, a close relative of random processes. The simplest of such systems is a particle moving on a line under the influence of a harmonic force and a friction:

$$\ddot{x} + 2k\dot{x} + bx = 0, \tag{1.1}$$

where: $x = x(t)$, $x \in \mathbf{R}^1$, is the position of the particle; k and b are constants; and overdot denotes the time-derivative. Clearly, for $k \neq 0$ the equation (1.1) is not a Hamiltonian (or a Lagrangian) system *as it stands*. However, set

$$x(t) = X(t)\, \mathrm{e}^{-kt}. \tag{1.2}$$

Then

$$\ddot{x} + k\dot{x} + bx = \left[\ddot{X} + \left(b - k^2\right) X\right] \mathrm{e}^{-kt} \tag{1.3}$$

and we get for X the equation

$$\ddot{X} + \left(b - k^2\right) X = 0 \tag{1.4}$$

which is a Hamiltonian system with the Hamiltonian

$$H(p, X) = \frac{p^2}{2} + \frac{\left(b - k^2\right)}{2} X^2.$$

The main theme of this paper is that some (not all!) random evolutions of Poisson type can be put into a Hamiltonian form after an appropriate time-dependent rescaling of the basic variables has been made. This rescaling into a Hamiltonian form is, naturally, a heuristic principle and not a general theorem. We shall see below how this principle works and fails to work for a few representative systems. We start in the next Section with one-dimensional random walk. Multidimensional generalizations proceed in two different directions: abstract telegraphers's equations, Section 3, or random walks in $\mathbf{R}^d$ and $\mathbf{Z}^d$, Section 4.

§ 2. One-Dimensional Random Walk

Let us consider a particle which moves on $\mathbf{R}^1$ with a constant speed v, and reverses direction according to a Poisson process with intensity a. This model had been proposed by G.I. Taylor [1] in an attempt to understand turbulent diffusion. It is convenient to work with a discrete situation first, and then pass to the continuous limit; the reader will find a lucid analysis in Kac [2] whose treatment I follow. So, suppose we have a lattice $\mathbf{Z}\Delta x$. Our particle moves with the speed v in the positive or negative direction; after time $\Delta t = \Delta x / v$ it changes direction; with probability $1 - a\Delta t$ the direction stays the same. Denote by S_n the displacement of the particle after n steps, with the initial step taken in the positive direction. Given a function $\varphi = \varphi(x)$, define the expectation values

$$F_n^+(x) := \langle \varphi(x + S_n) \rangle, \tag{2.1a}$$

$$F_n^-(x) := \langle \varphi(x - S_n) \rangle. \tag{2.1b}$$

Thus, $F_n^+(x)$ (resp. $F_n^-(x)$) is the expectation value of $\varphi(x)$ after n steps, when the initial direction of the walk starting at x is positive (resp. negative). Considering the n^{th} step as once removed from the $(n-1)^{st}$ one, we get in the usual way

$$F_n^{\pm}(x) = (1 - a\Delta t) F_{n-1}^{\pm}(x \pm v\Delta t) + a\Delta t F_{n-1}^{\mp}(x \pm v\Delta t) \tag{2.2}$$

which can be suggestively rewritten as

$$\begin{aligned} \frac{F_n^{\pm}(x) - F_{n-1}^{\pm}(x)}{\Delta t} &= \frac{F_{n-1}^{\pm}(x \pm v\Delta t) - F_{n-1}^{\pm}(x)}{\Delta t} \\ &\quad + a \left[F_{n-1}^{\mp}(x \pm v\Delta t) - F_{n-1}^{\pm}(x \pm v\Delta t) \right]. \end{aligned} \tag{2.3}$$

Passing to the continuous limit, we obtain

$$\begin{cases} \dfrac{\partial F^+}{\partial t} = v \dfrac{\partial F^+}{\partial x} - a(F^+ - F^-), \\ \dfrac{\partial F^-}{\partial t} = -v \dfrac{\partial F^-}{\partial x} - a(F^- - F^+). \end{cases} \tag{2.4}$$

This is the dynamical system we were after. The point of going through the discrete route first is the logical ease of deriving equation (2.2) (and similar equations later on). Now comes the rescaling. Set

$$F^{\pm} = f^{\pm} \mathrm{e}^{-at}. \tag{2.5}$$

Then the system (2.4) becomes

$$\begin{cases} \dfrac{\partial f^+}{\partial t} = v\dfrac{\partial f^+}{\partial x} + af^- \\ \dfrac{\partial f^-}{\partial t} = -v\dfrac{\partial f^-}{\partial x} + af^+ \end{cases} \tag{2.6}$$

and this is patently a Hamiltonian system since it can be written in the form

$$\frac{\partial}{\partial t}\begin{pmatrix} f^+ \\ f^- \end{pmatrix} = \begin{pmatrix} v\partial & -a \\ a & v\partial \end{pmatrix}\begin{pmatrix} \delta H/\delta f^+ \\ \delta H/\delta f^- \end{pmatrix} \tag{2.7}$$

with

$$H = \frac{(f^+)^2 - (f^-)^2}{2} \tag{2.8}$$

and with

$$\partial := \partial/\partial x. \tag{2.9}$$

The matrix

$$\begin{pmatrix} v\partial & -a \\ a & v\partial \end{pmatrix} \tag{2.10}$$

is skewsymmetric constant-coefficient and is, thus, Hamiltonian. See, e.g., [3], Ch. I, for the modern point of view on Hamiltonian formalism; all the Hamiltonian matrices below are of this simple kind. Note that the Hamiltonian H (2.8) is the first in the infinite series

$$H_n := \frac{f^+\partial^{2n}(f^+) - f^-\partial^{2n}(f^-)}{2}, \qquad n \in \mathbf{Z}_+ \tag{2.11}$$

of conserved densities of the system (2.6). Indeed, writing

$$h_1 \sim h_2 \tag{2.12}$$

when

$$(h_1 - h_2) \in Im\partial, \tag{2.13}$$

we have

$$\begin{aligned} \frac{\partial H_n}{\partial t} &\sim \frac{\delta H_n}{\delta f^+}\,\frac{\partial f^+}{\partial t} + \frac{\delta H_n}{\delta f^-}\,\frac{\partial f^-}{\partial t} \\ &= \partial^{2n}(f^+)\left[v\partial(f^+) + af^-\right] - \partial^{2n}(f^-)\left[-v\partial(f^-) + af^+\right] \\ &\sim \partial^{2n}(f^+)af^- - \partial^{2n}(f^-)af^+ \sim 0. \end{aligned}$$

Thus, H_n is a conserved density. Moreover, it is obvious that all the H_n's are in involution;

$$\{H_n, H_N\} := X_{H_n}(H_N) \sim 0, \qquad \forall\, n, N \in \mathbf{Z}_+ \tag{2.14}$$

where X_{H_n} is the evolution derivation corresponding to the flow with the Hamiltonian H_n:

$$\begin{aligned} X_{H_n}\begin{pmatrix} f^+ \\ f^- \end{pmatrix} &= \frac{\partial}{\partial t}\begin{pmatrix} f^+ \\ f^- \end{pmatrix} = \begin{pmatrix} v\partial & -a \\ a & v\partial \end{pmatrix}\begin{pmatrix} \delta H_n/\delta f^+ \\ \delta H_n/\delta f^- \end{pmatrix} \\ &= \begin{pmatrix} v\partial^{2n+1}(f^+) + a\partial^{2n}(f^-) \\ -v\partial^{2n+1}(f^-) + a\partial^{2n}(f^+) \end{pmatrix}. \end{aligned} \tag{2.15}$$

Thus, we have an infinite number of commuting flows with an infinity of commuting conserved densities.

We have considered the simplest possible system. Before moving on to more general pastures, it is worthwhile to note that the same equations (2.2) arise for the pair of functions, $p^+(x,t)$ and $p^-(x,t)$, describing the probability of finding the particle at the point x at the time t, arriving there from the right (for p^+) or left (for p^-) (see [4], Ch. I):

$$p^\pm(x, t + \Delta t) = (1 - a\Delta t)p^\pm(x \pm v\Delta t, t) + a\Delta t p^\mp(x \pm vt, t). \tag{2.16}$$

In this form this equation is easy to generalize for the inhomogeneous case and even for the case when the particle is allowed to rest (see [4], Ch. I):

$$\begin{aligned} &p^\pm(x, t + \Delta t) \\ &\quad = [1 - \sigma(x)]p^\pm(x,t) + [\sigma(x \pm v\Delta t) - \lambda(x \pm v\Delta t)\Delta t]p^\pm(x \pm v\Delta t, t) \\ &\quad + \lambda(x \pm v\Delta t)\Delta t p^\mp(x \pm v\Delta t, t), \end{aligned} \tag{2.17}$$

where $\lambda(x)$ is the local intensity of the Poisson process, and $1-\sigma(x)$ is the local probability of resting. Passing to the continuous limit we get

$$\begin{cases} \dfrac{\partial p^+}{\partial t} = v\dfrac{\partial}{\partial x}(\sigma(x)p^+) + \lambda(x)(p^- - p^+) \\ \dfrac{\partial p^-}{\partial t} = -v\dfrac{\partial}{\partial x}(\sigma(x)p^-) + \lambda(x)(p^+ - p^-). \end{cases} \tag{2.18}$$

If λ (formerly a) is not a constant, we cannot renormalize the variables $p^\pm$ by $\mathrm{e}^{-\lambda t}$ since t will enter *explicitly* into the motion equations; the system (2.18) in this case cannot be converted into a Hamiltonian form. When, however, λ is a constant, even though σ (formerly 1) is not, a Hamiltonian form is possible. Set

$$p^\pm = \widetilde{p}^{\,\pm}\mathrm{e}^{-\lambda t}. \tag{2.19}$$

Then the system (2.18) becomes

$$\begin{aligned} \frac{\partial \widetilde{p}^+}{\partial t} &= v\frac{\partial}{\partial x}(\sigma\widetilde{p}^+) + \lambda\widetilde{p}^- \\ \frac{\partial \widetilde{p}^-}{\partial t} &= -v\frac{\partial}{\partial x}(\sigma\widetilde{p}^-) + \lambda\widetilde{p}^+ \end{aligned} \tag{2.20}$$

which can be rewritten as

$$\frac{\partial}{\partial t}\begin{pmatrix} \widetilde{p}^{+} \\ \widetilde{p}^{-} \end{pmatrix} = \begin{pmatrix} v\partial & -\lambda\sigma^{-1} \\ \lambda\sigma^{-1} & v\partial \end{pmatrix}\begin{pmatrix} \delta H/\delta\widetilde{p}^{+} \\ \delta H/\delta\widetilde{p}^{-} \end{pmatrix} \tag{2.21}$$

with

$$H = \sigma\frac{(\widetilde{p}^{+})^2 - (\widetilde{p}^{-})^2}{2}. \tag{2.22}$$

§ 3. Telegrapher's Equation

The system (2.4) is 2-component first-order in time. "Now the amazing thing is that these two linear equations of first order can be combined into a [single] hyperbolic equation", says Kac ([2], p. 500), and proceeds as follows. Set

$$F := \frac{1}{2}(F^{+} + F^{-}), \qquad G := \frac{1}{2}(F^{+} - F^{-}) \tag{3.1}$$

so that

$$\frac{\partial F}{\partial t} = v\frac{\partial G}{\partial x} \tag{3.2a}$$

$$\frac{\partial G}{\partial t} = v\frac{\partial F}{\partial x} - 2aG \tag{3.2b}$$

whence

$$\frac{\partial^2 F}{\partial t^2} + 2a\frac{\partial F}{\partial t} = v^2\frac{\partial^2 F}{\partial x^2} \tag{3.3}$$

which is the telegrapher's equation. Rewritten as

$$v^{-2}\frac{\partial^2 F}{\partial t^2} + \frac{2a}{v^2}\frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial x^2} \tag{3.4}$$

it can be considered as a singular perturbation of the diffusion equation

$$\frac{1}{D}\frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial x^2} \tag{3.5}$$

where

$$\frac{1}{D} = \lim\frac{2a}{v^2}$$

when both a and v tend to infinity. Now, the diffusion equation assumes unlimited speeds of microscopic agents, clearly an untenable thesis in view of special relativity. The hyperbolic equation (3.4)can be considered then as a sort of relativistic regularization of the classical diffusion and heat equations. Let us now look at Hamiltonian properties of this equation, but first we generalize it to the form

$$\epsilon\frac{\partial^2 u}{\partial t^2} + 2a\frac{\partial u}{\partial t} = L(u) \tag{3.6}$$

where L is an arbitrary linear selfadjoint operator in arbitrary number of space dimensions:

$$L^{\dagger} = L \tag{3.7}$$

and ϵ is a constant (considered small if desired). The case

$$L = A^2 \tag{3.8}$$

where A is a skewadjoint operator:

$$A^{\dagger} = -A \tag{3.9}$$

is the most direct generalization of differential equations of telegrapher's type to which probabilistic interpretation applies [5]; more about this case later on. Set

$$u = \overline{u}\,\mathrm{e}^{\lambda t} \tag{3.10}$$

where λ is a constant to be specified presently. Since

$$\mathrm{e}^{-\lambda t}\left(\epsilon\frac{\partial^2}{\partial t^2} + 2a\frac{\partial}{\partial t}\right)\mathrm{e}^{\lambda t} = \epsilon\frac{\partial^2}{\partial t^2} + 2(\lambda\epsilon + a)\frac{\partial}{\partial t} + (\epsilon\lambda^2 + 2a\lambda),$$

choosing

$$\lambda = -a/\epsilon \tag{3.11}$$

we transform equation (3.6) into equation

$$\epsilon\frac{\partial^2\overline{u}}{\partial t^2} = \widehat{L}(\overline{u}), \tag{3.12}$$

where

$$\widehat{L} := L + \frac{a^2}{\epsilon} \tag{3.13}$$

is again a selfadjoint operator. The second-order equation (3.12), written as a first-order system

$$\begin{cases} \dfrac{\partial\overline{u}}{\partial t} = \widetilde{u} \\ \dfrac{\partial\widetilde{u}}{\partial t} = \epsilon^{-1}\widehat{L}(\overline{u}) \end{cases} \tag{3.14}$$

is easily seen to be a canonical Hamiltonian system:

$$\frac{\partial}{\partial t}\begin{pmatrix} \overline{u} \\ \widetilde{u} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \delta H/\delta\overline{u} \\ \delta H/\delta\widetilde{u} \end{pmatrix} \tag{3.15}$$

with

$$H = H_0 = \frac{\widetilde{u}^2}{2} - \frac{1}{2\epsilon}\overline{u}\widehat{L}(\overline{u}). \tag{3.16}$$

(It is in this place that the selfadjointness of $\widehat{L}$ plays a rôle). Like for the system (2.6), we have an infinity of commuting conserved densities for the system (3.14):

$$H_n = \frac{1}{2}\widetilde{u}\widehat{L}^n(\widetilde{u}) - \frac{1}{2\epsilon}\overline{u}\widehat{L}^{n+1}(\overline{u}), \qquad n \in \mathbf{Z}_+. \tag{3.17}$$

The alert reader may have noticed that the Hamiltonian form (2.7) of 1-dimensional random walk (2.4) is *different* from the canonical Hamiltonian form (3.15) of its generalization (3.6). How could this have happenend? The ultimate reason is that the *system* (2.4) is more rigid than the *scalar* second-order equation (3.6): the latter can be written in a multitude of ways as a 2-component first order system. For example, a direct generalization of the Hamiltonian form (2.7) exists for the case when $L = A^2$ with a skewadjoint A. Then equation (3.12) results from the following Hamiltonian system:

$$\begin{pmatrix} \dfrac{\partial \overline{u}}{\partial t} \\ \dfrac{\partial \widetilde{u}}{\partial t} \end{pmatrix} = \begin{pmatrix} X(\overline{u}) + \dfrac{a}{\epsilon}\widetilde{u} \\ -X(\widetilde{u}) + \dfrac{a}{\epsilon}\overline{u} \end{pmatrix} = \begin{pmatrix} X & -a\epsilon^{-1} \\ a\epsilon^{-1} & X \end{pmatrix} \begin{pmatrix} \overline{u} \\ -\widetilde{u} \end{pmatrix} \tag{3.18}$$

$$= \begin{pmatrix} X & -a\epsilon^{-1} \\ a\epsilon^{-1} & X \end{pmatrix} \begin{pmatrix} \delta/\delta\overline{u} \\ \delta/\delta\widetilde{u} \end{pmatrix} \left(\frac{\overline{u}^2 - \widetilde{u}^2}{2} \right) \tag{3.19}$$

where

$$X = \pm A\epsilon^{-1/2}. \tag{3.20}$$

Again,

$$H_n = \frac{1}{2}\overline{u}X^{2n}(\overline{u}) - \frac{1}{2}\widetilde{u}X^{2n}(\widetilde{u}), \qquad n \in \mathbf{Z}_+, \tag{3.21}$$

is an infinite commuting set of conserved densities of the system (3.18). In addition, the same equation (3.12) results from the following Hamiltonian system, quite *different* from (3.18):

$$\begin{pmatrix} \dfrac{\partial \overline{u}}{\partial t} \\ \dfrac{\partial \widetilde{u}}{\partial t} \end{pmatrix} = \begin{pmatrix} X(\widetilde{u}) + \dfrac{a}{\epsilon}\overline{u} \\ X(\overline{u}) - \dfrac{a}{\epsilon}\widetilde{u} \end{pmatrix} = \begin{pmatrix} X & a\epsilon^{-1} \\ -a\epsilon^{-1} & X \end{pmatrix} \begin{pmatrix} \widetilde{u} \\ \overline{u} \end{pmatrix} \tag{3.22}$$

$$= \begin{pmatrix} X & a\epsilon^{-1} \\ -a\epsilon^{-1} & X \end{pmatrix} \begin{pmatrix} \delta/\delta\overline{u} \\ \delta/\delta\widetilde{u} \end{pmatrix} (\overline{u}\widetilde{u})\,. \tag{3.23}$$

In this case, an infinity of commuting conserved densities is given by the formula

$$H_n = \overline{u}X^{2n}(\widetilde{u}), \qquad n \in \mathbf{Z}_+. \tag{3.24}$$

§ 4. Multidimensional Random Walk

A particle moves in $\mathbf{R}^d$ with a constant velocity $\boldsymbol{v} \in \{\boldsymbol{v}_i\}$. After each time interval Δt, there is a change of velocity. The change from $\boldsymbol{v}_i$ to $\boldsymbol{v}_j$ has the probability p_{ij}, and we take

$$p_{ij} = \delta_{ij} + \alpha_{ij}\Delta t, \tag{4.1}$$

with

$$\sum_j \alpha_{ij} = 0, \qquad \forall\, i. \tag{4.2}$$

(In the continuous limit, for the set of states $\{\boldsymbol{v}_i\}$ we have a Markov process $\xi(t)$ with the transition probabilities

$$p_{ij}(\Delta t) = \delta_{ij} + \alpha_{ij}\Delta t + \mathrm{o}(\Delta t), \tag{4.3}$$

but, as in § 1, it is more convenient to start with the discrete picture.) Denoting by $F_i = F_i(\boldsymbol{x}, t)$ the probability of finding the particle coming for its velocity change into the point $\boldsymbol{x}$ at time t with the velocity $\boldsymbol{v}_i$, we have, similar to § 1,

$$F_i(\boldsymbol{x}, t + \Delta t) = \sum_j p_{ji} F_j(\boldsymbol{x} - \boldsymbol{v}_i \Delta t, t). \tag{4.4}$$

By virtue of formula (4.1), in the continuous limit we get

$$\frac{\partial F_i}{\partial t} = -(\boldsymbol{v}_i \cdot \boldsymbol{\nabla})(F_i) + \sum_j \alpha_{ji} F_j \tag{4.5}$$

where

$$\boldsymbol{v}_i \cdot \boldsymbol{\nabla} := \sum_{s=1}^{d} (v_i)_s \frac{\partial}{\partial x_s}. \tag{4.6}$$

Set

$$F_i = f_i\, \mathrm{e}^{-\lambda t}. \tag{4.7}$$

Then equation (4.5) becomes

$$\frac{\partial f_i}{\partial t} = -(\boldsymbol{v}_i \cdot \boldsymbol{\nabla})(f_i) + \sum_j \beta_{ji} f_j \tag{4.8}$$

where

$$\beta_{ji} := \alpha_{ji} + \lambda \delta_{ij} \tag{4.9}$$

so that the constraint (4.2) turns into

$$\sum_j \beta_{ij} = \lambda, \qquad \forall\, i. \tag{4.10}$$

We are going to analyze the system (4.8), (4.10) from the Hamiltonian point of view. As the Hamiltonian we pick

$$H = \frac{1}{2} \sum_i c_i (f_i)^2 \tag{4.11}$$

with some unknown constants c_i's. The constant-coefficient Hamiltonian matrix

$$B_{ij} = -\delta_{ij} \frac{1}{c_i} \boldsymbol{v}_i \cdot \boldsymbol{\nabla} + \Gamma_{ij} \tag{4.12}$$

where $\Gamma = (\Gamma_{ij})$ is a constant skewsymmetric matrix, reproduces the motion equations (4.8) through the Hamiltonian ansatz

$$\frac{\partial f_i}{\partial t} = \sum_j B_{ij} \left(\frac{\delta H}{\delta f_j} \right) \tag{4.13}$$

iff

$$\beta_{ji} = \Gamma_{ij} c_j \qquad \text{(no sum on } j\text{)}. \tag{4.14}$$

Let us estimate the proportion of Hamiltonian random walks among all of them. The dimension of the latter is the dimension of the space of the β's subject to the conditions (4.10), but with the understanding that λ is at our disposal. Thus,

$$\text{Total dim } = N^2 - N + 1 \tag{4.15}$$

where N is the number of the different f_i's (and also the number of the velocities $\boldsymbol{v}_i$'s). From (4.10) and (4.14) we get

$$\lambda = \sum_i \beta_{ji} = \sum_i \Gamma_{ij} c_j = \left(\sum_i \Gamma_{ij} \right) c_j$$

so that

$$c_j = \frac{\lambda}{\sum_i \Gamma_{ij}} \tag{4.16}$$

(or no conditions for $\lambda = 0$). Thus, we have to look at the dimension of the image of the map $\Gamma \times \boldsymbol{c} \longmapsto (\Gamma \widehat{c})^t = \beta$, where

$$\boldsymbol{c} := (c_1, \ldots, c_N)^t, \qquad \widehat{c} := \text{ diag } (c_1, \ldots, c_N), \tag{4.17}$$

and where $\boldsymbol{c}$ is a function of Γ given by formula (4.16). Let us compute this dimension at the point in the β-space which corresponds to

$$\Gamma_0 = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix}; \tag{4.18}$$

thus, we assume that N is even:

$$N = 2\overline{N}. \tag{4.19}$$

Then

$$\widehat{c}_0 = \lambda \begin{pmatrix} -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \tag{4.20}$$

and

$$\beta_0^t = \lambda \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}. \tag{4.21}$$

(This β corresponds to a direct sum of one-dimensional random walks.) Let

$$\Gamma = \Gamma_0 + \epsilon\overline{\Gamma} \tag{4.22}$$

$$c = c_0 + \epsilon\overline{c} \tag{4.23}$$

$$\beta^t = \beta_0^t + \epsilon\overline{\beta}^t \tag{4.24}$$

be an infinitesimal change of the objects under consideration (i.e., $\epsilon^2 = 0$). Denote

$$\gamma_j := \sum_i \overline{\Gamma}_{ij} \tag{4.25}$$

$$\overline{\Gamma} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ -\mathcal{B}^t & \mathcal{C} \end{pmatrix}, \qquad \mathcal{A}^t = -\mathcal{A}, \quad \mathcal{C}^t = -\mathcal{C}. \tag{4.26}$$

Then

$$\widehat{c} = \lambda \begin{pmatrix} -\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} - \lambda\epsilon \text{ diag } (\gamma_1, \ldots, \gamma_N) \tag{4.27}$$

and hence

$$\lambda^{-1}\overline{\beta}^t = \begin{pmatrix} & & & -\gamma_{\overline{N}+1} & & \\ & & & & \ddots & \\ & & & & & -\gamma_{2\overline{N}} \\ \gamma_1 & & & & & \\ & \ddots & & & & \\ & & \gamma_{\overline{N}} & & & \end{pmatrix} + \begin{pmatrix} -\mathcal{A} & \mathcal{B} \\ \mathcal{B}^t & \mathcal{C} \end{pmatrix}. \tag{4.28}$$

Taking $\mathcal{A}$ and $\mathcal{C}$ out of $\lambda^{-1}\overline{\beta}^t$, we are left with the matrix

$$\mathcal{B} - \text{ diag } \left(\sum_i \mathcal{B}_{i1}, \ldots, \sum_i \mathcal{B}_{i\overline{N}} \right) \tag{4.29}$$

which amounts to an arbitrary matrix $\widehat{\mathcal{B}}$ subject to the conditions

$$\sum_i \widehat{\mathcal{B}}_{ij} = 0, \qquad \forall\ j. \tag{4.30}$$

Thus, the dimension of the $\overline{\beta}$'s is

$$\frac{\overline{N}^2-\overline{N}}{2}+\frac{\overline{N}^2-N}{2}+(\overline{N}^2-\overline{N})=2(\overline{N}^2-\overline{N})=\frac{N^2}{2}-N. \tag{4.31}$$

Taking into account our free parameter λ, we finally get the dimension of the space of Hamiltonian random walks around the point β_0:

$$\text{Ham dim }=\frac{N^2}{2}-N+1 \tag{4.32}$$

which is more than half the total dimension (4.15) of the space of random walks. I conclude with a few remarks.

Remark 4.33. Even for $d=1$ the random walk model in this section is more general than the one-dimensional model considered in § 1, since we allow $N(=2\overline{N})$ different velocities rather than two. The case $d=1$ is special, having all possible velocities being proportional to each other. This fact leads to an existence of an infinity of commuting conserved densities for the system (4.8):

$$H_n=\frac{1}{2}\sum_i c_i[\partial^n(f_i)]^2. \tag{4.34}$$

Remark 4.35. In our calculations of dimensions we had no use for the convective terms $(\boldsymbol{v}_i\cdot\boldsymbol{\nabla})(f_i)$. Had these terms been absent to begin with, e.g., for $\boldsymbol{x}$-independent solutions, we would have been dealing with a system of ordinary differential equations for the f_i's; formula (4.32) in this case provides a (very) low bound on the dimension of such systems with a particular Hamiltonian representation. Such systems

$$\dot{\boldsymbol{F}}^t=\boldsymbol{F}^t\beta \tag{4.36}$$

where

$$\boldsymbol{F}^t:=(F_1,\dots,F_N),\qquad \beta:=(\beta_{ij})$$

appear, e.g., for column-sums of the inverse Kolmogorov equation for a Markov process with transition probabilities (4.3):

$$\frac{d}{dt}\,p_{ij}=\sum_k p_{ik}\beta_{kj} \tag{4.37}$$

where

$$\beta_{kj}:=\alpha_{kj}+\lambda_j\delta_{kj} \tag{4.38}$$

and

$$F_j:=\sum_i p_{ij}. \tag{4.39}$$

Hamiltonian analysis can be applied to the full system (4.37) and also to the direct Kolmogorov equation

$$\frac{d}{dt}\,p_{ij}=\sum_k \beta_{ik}p_{kj}. \tag{4.40}$$

Remark 4.41. If the set of all possible velocities $\{\boldsymbol{v}_i\}$ is not discrete but is a continuous one, the sum sign in equation (4.8) turns into an integral sign. The Hamiltonian arguments undergo a similar minor modification.

Remark 4.42. If the randomly walking particle has internal degrees of freedom [6], equation (4.4) changes into

$$F_i^{\mu}(\boldsymbol{x}, t+\Delta t) = \sum p_{ji}^{\nu\mu} F_j^{\nu}(\boldsymbol{x} - \boldsymbol{v}_i \Delta t, t) \tag{4.43}$$

with

$$p_{ij}^{\mu\nu} = \delta_{ij}\delta_{\mu\nu} + \alpha_{ij}^{\mu\nu}\Delta t \tag{4.44}$$

where indices μ, ν refer to the internal states. Equation (4.5) then becomes

$$\frac{\partial F_i^{\mu}}{\partial t} = -(\boldsymbol{v}_i \cdot \boldsymbol{\nabla})(F_i^{\mu}) + \sum_{j,\nu} \alpha_{ji}^{\nu\mu} F_j^{\nu} \tag{4.45}$$

with

$$\sum_{i,\mu} \alpha_{ji}^{\nu\mu} = 0, \qquad \forall\, j, \nu. \tag{4.46}$$

The Hamiltonian analysis proceeds as before, with the quadratic Hamiltonian now being

$$H = \frac{1}{2} \sum_{i,\mu,\nu} c_i^{\mu\nu} F_i^{\mu} F_i^{\nu}. \tag{4.47}$$

In the simplest one-dimensional case, where $\{\boldsymbol{v}_i\} = \{\pm \boldsymbol{v}\}$, and where everything is invariant with respect to the reflection $x \mapsto -x$, as in § 1, we have

$$\begin{cases} \dfrac{\partial \boldsymbol{F}_+}{\partial t} = v\dfrac{\partial \boldsymbol{F}_+}{\partial x} + \widetilde{\alpha}\boldsymbol{F}_+ + \overline{\alpha}\boldsymbol{F}_- \\ \dfrac{\partial \boldsymbol{F}_-}{\partial t} = -v\dfrac{\partial \boldsymbol{F}_-}{\partial x} + \widetilde{\alpha}\boldsymbol{F}_- + \overline{\alpha}\boldsymbol{F}_+ \end{cases} \tag{4.48}$$

where

$$(\boldsymbol{F}_{\pm})^{\mu} = (F_{\pm}^{\mu}), \qquad \widetilde{\alpha}^{\mu\nu} := \alpha_{++}^{\nu\mu} = \alpha_{--}^{\nu\mu}, \qquad \overline{\alpha}^{\mu\nu} := \alpha_{+-}^{\nu\mu} = \alpha_{-+}^{\nu\mu},$$

etc. We also have a vector analog of the telegrapher's equation: Set

$$\boldsymbol{F} := \frac{1}{2}(\boldsymbol{F}_+ + \boldsymbol{F}_-), \qquad \boldsymbol{G} := \frac{1}{2}(\boldsymbol{F}_+ - \boldsymbol{F}_-) \tag{4.49}$$

so that

$$\begin{cases} \dfrac{\partial \boldsymbol{F}}{\partial t} = v\dfrac{\partial \boldsymbol{G}}{\partial x} + (\widetilde{\alpha} + \overline{\alpha})\boldsymbol{F} \\ \dfrac{\partial \boldsymbol{G}}{\partial t} = v\dfrac{\partial \boldsymbol{F}}{\partial x} + (\widetilde{\alpha} - \overline{\alpha})\boldsymbol{G}. \end{cases} \tag{4.50}$$

Then

$$\frac{\partial^2 \boldsymbol{F}}{\partial t^2} - 2\widetilde{\alpha}\frac{\partial \boldsymbol{F}}{\partial t} = v^2 \frac{\partial^2 \boldsymbol{F}}{\partial x^2} + \left(\overline{\alpha}^2 - \widetilde{\alpha}^2 + [\overline{\alpha}, \widetilde{\alpha}]\right) \boldsymbol{F} \tag{4.51}$$

where the matrices $\widetilde{\alpha}$ and $\overline{\alpha}$ are subject to the condition

$$(1, \ldots, 1)(\widetilde{\alpha} + \overline{\alpha}) = (0, \ldots, 0). \tag{4.52}$$

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